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(Residential Autonomous College under University of Calcutta)

B.A./B.Sc. THIRD SEMESTER EXAMINATION, DECEMBER 2014

SECOND YEAR

MATHEMATICS (Honours)

Paper : III

Date : 17/12/2014
Time : 11 am – 3 pm
Full Marks : 100

[Use a separate Answer Book for each group]

Group – A

(*Answer* ***any five*** *questions*) [5×10]

1. a) Prove that every square matrix satisfies its own characteristic equation. [5]

   b) Diagonalize the following matrix orthogonally [5]

$$A=\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$$

2. a) Let V be a finite dimensional vector space and W be a subspace of V. Prove that $\dim\frac{V}{W}=\dim V-\dim W$. [5]

   b) For which values of 'a' the system of equations is consistent? Solve it completely, in case it is consistent. [5]

$$x-y+z=1,\ x+2y+4z=a,\ x+4y+6z=a^2$$

3. a) If α,β be any 2 vectors in a Euclidean Space V, then prove that $\|\alpha+\beta\|\le\|\alpha\|+\|\beta\|$. [2]

   b) If A is a real symmetric matrix, then prove that there's a real orthogonal matrix P such that $P^{-1}AP$ is diagonal. [4]

   c) The matrix of a linear mapping $T:\mathbb{R}^3\to\mathbb{R}^2$ relative to the ordered bases $\{(0,1,1),(1,0,1),(1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0),(1,1)\}$ of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find the matrix of T relative to the ordered bases $\{(1,1,0),(1,0,1),(0,1,1)\}$ of $\mathbb{R}^3$ and $\{(1,1),(0,1)\}$ of $\mathbb{R}^2$. [4]

4. a) Using rank-nullity theorem show that the linear transformation $T:P_2(\mathbb{R})\to P_3(\mathbb{R})$ given by $T(f(x))=2f'(x)+\int_0^x 3f(t)dt$ is one-to-one. [2]

   b) A linear transformation $T:P_2(\mathbb{R})\to M(2,\mathbb{R})$ is defined by $T(f(x))=\begin{pmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{pmatrix}$. Find rank (T). [3]

   c) Are the following matrices given below, similar? Justify.

$$A=\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix},\ B=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$ [5]

5. a) Transform the quadratic form

$$f(x_1,x_2,x_3)=(x_1\ x_2\ x_3)\begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

   to normal form by elementary transformations and give the coordinate transformation **x = cy**. [5]

b) Prove that, each eigen value of a real orthogonal matrix has unit modulus. [5]

6. a) Find the matrix representation of $D: P_3(\mathbb{R}) \to P_3(\mathbb{R})$ with respect to the standard ordered basis $B = \{1, x, x^2, x^3\}$ where, $D \equiv \frac{d}{dx}$. Find the eigen values of the matrix of D and corresponding eigen vectors as well. [2+2+1]

b) Let $V = P(\mathbb{R})$ with inner product $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)dt$, and consider the subspace $P_2(\mathbb{R})$ with standard ordered basis β. Use the Gram-Schmidt process to replace β by an orthogonal basis $\{v_1, v_2, v_3\}$ where $v_1 = 1$. Hence obtain an orthonormal basis for $P_2(\mathbb{R})$. [3+2]

7. a) Let, $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis of a Euclidean space V of dimension 'n'. Then, prove that a linear mapping $T: V \to V$ is orthogonal <u>if and only if</u> $\langle T\alpha_i, T\alpha_j\rangle = \langle \alpha_i, \alpha_j\rangle \; \forall \; 1 \le i, j \le n$. [5]

b) Given $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Find A^{50}. [2]

c) Prove that the eigen values of a real-symmetric matrix are all real. [3]

8. a) Find the orthogonal complement of the row space of the matrix [5]

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{pmatrix}$$

b) Let V be a vector space of dimension 'n' over a field F. Then prove that V is isomorphic to F^n. [5]

<u>Group – B</u>

(*Answer **<u>any four</u>** questions from **<u>Q.No. 9 - 15</u>***) [4×5]

9. Show that the origin lies in the acute angle between the planes $x + 2y + 2z = 9$, $4x - 3y + 12z + 13 = 0$. Find the planes bisecting the angles between them and point out which bisects the acute angle. [2+1+2]

10. If 2α be the angle between two skew-lines and 2c is the shortest distance between them then show that the equations of the lines can be expressed in the form $y = mx$, $z = c$ and $y = -mx$, $z = -c$, where $m = \tan\alpha$. [5]

11. A variable sphere passes through the points $(0, 0, \pm c)$ and cuts the straight lines $y = x\tan\alpha$, $z = c$ and $y = -x\tan\alpha$, $z = -c$ at the points P and P' other than $(0, 0, \pm c)$. If $PP' = 2a$ (a = constant), show that the centre of the sphere lies on the circle $x^2 + y^2 = (a^2 - c^2)\operatorname{cosec}^2 2\alpha$, $z = 0$. [5]

12. Find the reciprocal cone of the cone whose vertex is (0,0,d) and the base is $x^2 + y^2 - 2cx = 0$, $z = 0$. [5]

13. Prove that among all central conicoids, the only ruled surface is the hyperboloid of one sheet. [5]

14. Show that the perpendiculars from the origin to the generators of the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ lie on the surface $\frac{a^2(b^2+c^2)^2}{x^2} + \frac{b^2(c^2+a^2)^2}{y^2} = \frac{c^2(a^2-b^2)^2}{z^2}$. [5]

15. Reduce the equation $4x^2 - y^2 - z^2 + 2yz - 8x - 4y + 8z - 2 = 0$ to its canonical form and hence state the nature of the conicoid represented by it. [5]

(*Answer* ***any two*** *questions from* ***Q.No. 16 - 18***) [2×15]

16. a) Find the tangential and normal components of velocity and acceleration of a particle describing a plane curve. [6]

b) Over a small smooth pulley is placed a uniform flexible cord, the latter is initially at rest and lengths $\ell - a$ and $\ell + a$ hang down on the two sides. The pulley is now made to move with constant upward acceleration f. Show that the string will leave the pulley after a time $\sqrt{\frac{\ell}{f+g}} \cosh^{-1}\left(\frac{\ell}{a}\right)$. [6]

c) Find the law of force parallel to the axis of y under which a particle is moving along the curve $xy = a^2$ (a is constant). [3]

17. a) If h be the height attained by a particle when projected with a velocity V from the earth's surface supposing its attraction constant and H the corresponding height when the variation of gravity is taken into account, prove that $\frac{1}{h} - \frac{1}{H} = \frac{1}{r}$, where r is the radius of the earth. [5]

b) Find the law of force to the pole when the path is the cardioide $r = a(1 - \cos\theta)$ and prove that if F be the force at the apse and v the velocity there then $3v^2 = 4aF$. [5]

c) A particle describes a plane curve under the action of a central force P per unit mass. Prove that in usual notation, the differential equation of the path of the particle is $\frac{h^2}{p^3}\frac{dp}{dr} = P$. [5]

18. a) A particle describes an elliptic orbit under a force which is always directed towards the centre of the ellipse. Find the law of force. [6]

b) An engine working at a constant rate H draws a load M against a resistance R. Show the maximum speed is $\frac{H}{R}$ and that the time taken to attain half this speed is $\frac{MH}{R^2}\left(\log 2 - \frac{1}{2}\right)$. [4]

c) A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their attractions per unit of mass at unit distance being μ and μ'. The particle is displaced towards one of them. Show that its motion is oscillatory of period $\frac{2\pi}{\sqrt{\mu + \mu'}}$. [5]

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